Day 2: Problem Analysis

15.04.2015

Problem A. Another 2048 Problem

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Problem statement

- You are given multiset of integers
- In one turn you can take two equal numbers, remove these two numbers from the set and insert their sum instead
- You have to find the number of subsets, such that you can somehow get 2048 making several turns

Solution

- Sum up only powers of two, otherwise can't get 2048 only multiplying by 2
- Remove all m non-powers of two, and multiply the answer by 2^m
- Numbers don't exceed 2047, all the subsets with sum of their numbers $\geqslant 2048$ are good
- Calculate number of bad subsets $\Rightarrow$ knapsack problem
- $f[i][s]$ — the number of subsets, containing numbers up to 2^i with sum equal to s
- Optimize: one only needs $f[i][s]$, such that s is multiple of 2^i, otherwise round down

Problem B. Big Kingdom

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Problem statement

- Given n guards in points (x_i, y_i) on the plane, and velocity for every guard v_i
- The point is guarded by the i-th guard, if the time to get to this points is strictly less, than for any other guard
- Find if the area guarded by every guard is infinite

Solution

- If for some guard there exists faster than him, then his area is not infinite
- Get all the guards, build convex hull
- Guard on the border $\Leftrightarrow$ his guarded area is infinite

Problem C. Construct The Array

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Problem statement

- You have an array. Two kinds of operations made:
  1. Add v to all a_x, such that $\texttt{gcd}(x, n) = d$
  2. Calculate $\sum_{i=1}^{x} a_i$

Solution

- Suppose $d = 1$, $n = p_1^{w_1} \cdot p_2^{w_2} \cdot \ldots \cdot p_k^{w_k}$, then x shouldn't be divisible to all p_i
- Use inclusion-exclusion principle, add $(-1)^{|A|} \cdot v$ to $f\left[\prod_{i \in A} p_i\right]$
- $\sum_{i=1}^{x} a_i = \sum_{i=1}^{x} f[i] \cdot \left\lfloor \frac{x}{i} \right\rfloor$
- $d \neq 1$ is almost the same, supposing $n := \frac{n}{d}$ and $x = d \cdot \prod_{i \in A} p_i$
- Calculate $\sum_{i=1}^{x} f[i] \cdot \left\lfloor \frac{x}{i} \right\rfloor$ in $O(\sqrt{x})$, since there are about $\sqrt{x}$ different values of $\left\lfloor \frac{x}{i} \right\rfloor$

Problem D. Dictator

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Problem statement

- Given a tournament
- You need to arrange the vertices in order, such that for every previous vertex there is a path of length 1 or 2, passing only previous vertices

Solution

- There is always the answer
- Look at the vertex v, that has the most incoming edges k, make it last in the order
- Need to prove, that from every other vertex u there is the path of length 1 or 2
- Two cases:
  - $uv \in E$, then the edge is the path
  - Otherwise, there exists w, such that $uw \in E$ and $wv \in E$. Suppose it's not, then u has k incoming edges from first type of vertices and one incoming edge from v, making in total $k + 1$ edges. So k is not maximum number of incoming edges, leading to the contradiction

Problem E. Electricity and Magic

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Problem statement

- Given a grid, consisting of 0s and 1s
- You can make two types of operations:
  1. Flip bits that are adjacent to (i, j)
  2. Flip bits that are adjacent to (i, j) and (i, j) itself
- Find minimum number of moves needed to obtain a field filled with 0s

Solution

- Dynamic programming approach
- For every cell you need to remember one of three states:
  - You made moves on cell (i, j)
  - You didn't make any move on cell, it is equal to 0
  - You didn't make any move on cell, it is equal to 1
- If you made move on cell, then you can make it either 0 or 1, changing the type of one of the moves
- $O(nm3^m)$ solution

Problem F. Fight

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Problem statement

- Hero attacks monster reducing monster's HP h by a
- Monster HP increases by b every second, after the attack if it's been made
- Hero can't make $k+1$ consecutive attacks
- Can hero kill the monster?

Solution

- Killing in first round $\Leftrightarrow h \leqslant a$
- Killing before rest $\Leftrightarrow h + b(k-1) \leqslant ak$
- Killing with rest $\Rightarrow b(k+1) < ak$

Problem G. Go and restore!

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Problem statement

- Given multiplication table base p
- Digits were shuffled, and table was rearranged
- Find the permutation, that shuffles the digits

Solution

- Look at row which corresponds to $p-1$
- It looks like $[0][0]$, $[0][p-1]$, $[1][p-2]$, $[2][p-3] \ldots [p-2][1]$
- No other row has $p-2$ as highest digit, so it is the only row, that has $p-1$ different highest digits
- Find that row, learn what $p-1$ maps to
- Using the fact that $(p-1) \times k = [k-1][p-k]$, find $map[k-1]$ from $map[k]$

Problem H. Hidden Integer

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Problem statement

- Given integer x
- You make k moves
- During i-th move, you replace x by minimal y, such that $y \geqslant x$ and y is divisible by i
- Find the value of x after k moves

Solution

- Let's represent the value of x after i moves as $a \cdot i$
- After the move x will equal to $b \cdot (i+1)$ for some b, such that $b \cdot (i+1) \geqslant a \cdot i$
- So $b = \left\lceil \frac{a \cdot i}{i+1} \right\rceil$
- It can be proven, that if a stoped changing, then it never changes again, and this moment happens fast

Problem J. Just do it

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Problem statement

- You are to calculate permanent of matrix... Of m matrices.
- i-th of them:
  - If $x \neq y$, then $w_{x,y} = 1$
  - If $x = y$ and $x \leqslant n$, then $w_{x,y} = a_x$
  - If $x = y$ and $x > n$, then $w_{x,y} = 0$

Solution

- Let's solve for first matrix
- The permanent of this matrix equals to:
$$\sum_{k=0}^{n}\left(\sum_{|A|=k}\prod_{x\in A}a_x\right)\cdot f_{n-k}$$
- Where f_x — is the number of permutations of length x without fixed points
- Generally, the permanent for i-th matrix is
$$\sum_{k=0}^{n+i-1}\left(\sum_{|A|=k}\prod_{x\in A}a_x\right)\cdot f_{n+i-1-k}$$
- It's easy to see: $\sum_{k=0}^{n}\left(\sum_{|A|=k}\prod_{j\in A}a_j\right)x^k=\prod_{i=1}^{n}(1+a_ix)$

Solution

- To calculate $\prod_{i=1}^{n}(1+a_i x)$ use Fast Fourier Transform and Divide and Conquer technique
- Calculate $f_n = \sum_{i=0}^{n} \frac{(-1)^i n!}{i!}$
- Computing permanents for all matrices is also reduced to polynomial multiplication
- Solution time complexity: $O(n \log^2 n)$

Problem L. Light Source

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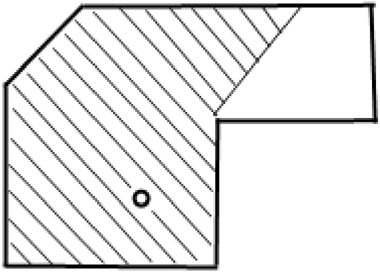
Problem statement

- You are given polygon and a point inside.
- Calculate the area visible from this point



Solution

- Use scanline technique rotating the ray from the given point
- Maintain the set of sides of polygon, that intersect this ray, sorted by distance between center and intersection
- When vertex is met, either the segment appears or disappears
- For angle between two events, add the triangle area to the answer