Geometry compilation contest

Problem analysis

Artem Vasilev   Vitaly Aksenov

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A. Totalphone

Problem statement

Find the maximal distance from the $(0, 0) - - - (L, 0)$ segment to n given points.

A. Totalphone

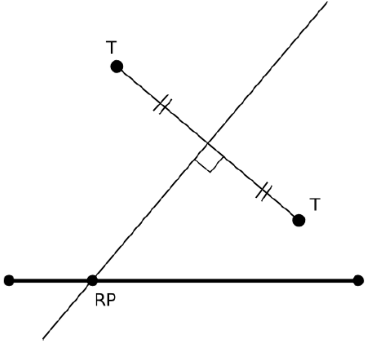
Simple solution

- Use binary search on the answer R.
- One point covers the intersection of line with the circle of radius R.
- Check that these intersections cover the line completely.
- $O(N \log N \log R)$.
- You may notice that you don't need to sort events, and the "check" part can be done in $O(N)$.
- $O(N \log R)$ solution.

A. Totalphone

Fact

- Two points split their area of influence (segment where this point is the closest one) by the perpendicular bisector.



- Each point could have only one segment of influence.

A. Totalphone

Solution

- We could use a "convex hull" trick for the points, process them from left to right.
- Add new point and pop points from stack until our new point "covers" their intervals.
- Finally, add new point on stack and calculate its interval.
- After all intervals are calculated, get answer in $O(N)$.

B. Rural planning

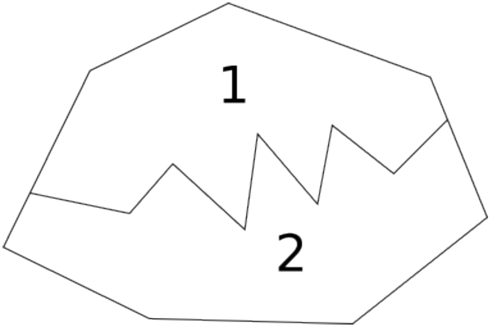
Problem statement

You need to find a polygon that contains all given points as vertices such that the area of the polygon is not smaller than the half of an area of the convex hull.

B. Rural planning

Solution

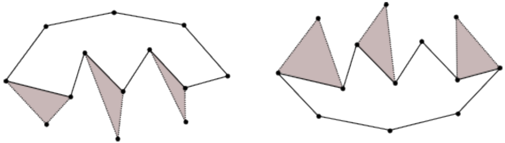
- Suppose we split the convex hull into 2 chains: lower and upper.
- Then, we take all the points inside and make an arbitrary polyline through them.



- Larger of these 2 polygons will have area greater than half of the convex hull.

B. Rural planning
Solution

- We can extend the larger polygon to use the points from the other hull.
- As these points are from convex hull, the area only increases if we add them.
- Go through points from the other hull and connect them to the nearest point to the left and to the right.



- Output the larger of two polygons.

C. Two circles

Problem statement

Draw two non-intersecting circles of maximum possible radius R inside the given convex polygon.

C. Two circles

Solution

- Run binary search on the answer R.
- Then shift all the sides of the polygon inside by R.
- Intersect new n half-planes. The lines are already sorted by angle, so this step is similar to Graham's algorithm for finding the convex hull.
- Find the diameter of the new polygon in linear time using rotating calipers.

D. Aerobics

Problem statement

- You have to place N circles with centers in the rectangle, such that they do not intersect.
- It is known that the area of the rectangle is 5 times bigger than the total area of circles.

D. Aerobics

Randomized solution

- Insert circles in the order of decreasing radius.
- For each circle we choose the random point and check if we can place a mat there.
- The total area of bad points does not exceed
  $\sum_{j=0}^{i-1} \pi(R_i + R_j)^2 \leq 4 \sum_{j=0}^{i-1} \pi R_j^2 \leq \frac{4}{5} S.$
- Thus with probability at least $\frac{1}{5}$ we find a good point.

- During each step you could rotate the current position 90 degrees clockwise around some of the given points.
- Find the maximal distance you could get from $(0, 0)$ after M steps.

E. Twirling Towards Freedom

Understating rotations

- Rotation around the point is the same as multiplying the vector by $-i$.
- $P_1 - Q = -i \cdot (P_0 - Q)$, thus $P_1 = -i \cdot P_0 + (1 - i)Q$.
- Suppose we rotate M times the starting point P_0:

$$\begin{aligned} P_n = -i \cdot P_0 + & (1 - i) \cdot (Q_{M-1} + Q_{M-5} + \ldots) + \\ & (-1 - i) \cdot (Q_{M-2} + Q_{M-6} + \ldots) + \\ & (-1 + i) \cdot (Q_{M-3} + Q_{M-7} + \ldots) + \\ & (1 + i) \cdot (Q_{M-4} + Q_{M-8} + \ldots). \end{aligned}$$

- For each direction v we want to choose the furthest point in direction $(1 - i) \cdot v$, which will replace $Q_{M-1}, Q_{M-5}, \ldots$.
- Similarly, for $Q_{M-2}, Q_{M-6}, \ldots$ in direction $(-1 - i) \cdot v$, for $Q_{M-3}, Q_{M-7}, \ldots$ in direction $(-1 + i) \cdot v$ and for $Q_{M-4}, Q_{M-8}, \ldots$ in direction $(1 + i) \cdot v$.
- This could be done by general method of rotating calipers.

F. Irregular cake

Problem statement

You are given a polygon. You have to split it onto g parts with equal areas using only vertical cuts.

F. Irregular cake

Solution

- Calculate the area of the polygon S.
- For each i we want to find a cut that produces the part with area $(i + 1)\frac{S}{g}$.
- Use binary search on the location of the cut.
- Calculate the area of the part of the polygon lying to the left of the cut.

G. Rope intranet

Problem statement

- You are given two lines with some number of points.
- There are some segments from the points on the first line to the points on the second.
- You have to count the number of intersections.

G. Rope intranet

Solution

- Not really a geometry problem.
- Take a segment and iterate through all other segments.
- Two segments intersect if $a_i < a_j$ and $b_i > b_j$.
- The same as the number of inversion between array a and b.

H. Watering plants

Problem statement

You have to cover given circles by two circles with minimal radius r.

H. Watering plants

Solution

- Binary search on the answer r.
- We can always move a circle in such a way that it will touch some two circles.
- We build all the candidate circles and check each pair of them to cover all given circles.

H. Watering plants

Find the large circle by two small circles

- We are given two circles with radius r_1 and r_2 and want to build a circle with radius r that touches them.
- The center of that circle is the point of intersection of two circles with radiuses $r - r_1$ and $r - r_2$ with the centers in the corresponding centers of two provided circles.

I. Pinball

Problem statement

- You are given a ball that falls from infinite height at $x = X$.
- There are some segments that could change trajectory of a ball.
- When the ball falls onto some segment, it slides off at its lowest point.
- Find last the x-coordinate of the ball.

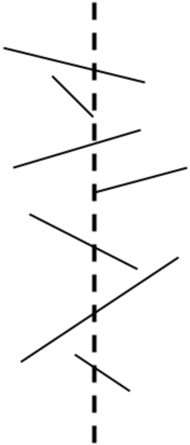
I. Pinball

Solution

- Let's for every segment find another segment right 'below' it
- After that's known, the rest of the problem is trivial
- Use sweep line technique
- Create 'left' and 'right' events for each segment, sort all events
- Keep currently open segments sorted by their y-coordinate.

I. Pinball

Solution



I. Pinball

Solution

- You can compare two segments given a current x-coordinate.
- It is possible to do it with a standard `vectMul` predicate.
- All the segments you have to compare have overlapping x-projections.
- Find any point on one segment that is in x-projection of another, and check the angle.