Combinatorics compilation contest

Problem analysis

Artem Vasilev   Vitaly Aksenov

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A. Scott's New Trick

Problem statement

- You are given two sequences a_i and b_i and a prime modulo m.
- You have to calculate the number of pairs i and j such that $a_i \cdot b_j \mod m \leq L$.

A. Scott's New Trick

Solution

- Let us find a generator g for modulo m.
- Represent a_i and b_i as g^{α_i} and g^{β_i}, i.e., by calculating how many from the sequences are equal g^i modulo m.
- Multiplication becomes addition: $a_i \cdot b_j = g^{\alpha_i + \beta_j}$
- Consider a_i and b_j as polynomials, calculate the convolution using FFT.
- After convolution go back to powers of g and get the result.

B. Cryptography Entertainment

Problem statement

Calculate the number of strings with hash code equal to x modulo $998244353 = 7 \cdot 17 \cdot 2^{23} + 1$.

B. Cryptography Entertainment
Solution

- $dp[i][j]$ — the number of strings with length i that have hash j modulo m.
- $dp[a+b][j] = \sum_{l=0}^{m} dp[a][l \cdot p^{-b}] \cdot dp[b][j-l]$.
- Consider $dp[i](x)$ as the polynomial with coefficients $dp[i][j]$.
- Equation above looks like the multiplication of $dp[b](x)$ and slightly modified $dp[a](x)$ ($dp[a] \to dp[a \cdot p^b]$).
- We could calculate all $dp[2^k](x)$ in $O(m \log m \log n)$ using FFT.
- Then we have to multiply $dp[2^k](x)$ corresponding to ones in n's binary representation.

C. Equal Sums

Problem statement

- You are given 500 integers not bigger than 10^{12}.
- You have to find two distinct subsets with the same sum.

C. Equal Sums

Randomized solution

- Leave only 50 integers (any of them).
- Since $2^{50} > 50 \cdot 10^{12}$ there exist two equal subsets.
- Take random 6 integers out of chosen 50.
- Add them up and store in hashset.
- Repeat until you find the answer.

C. Equal Sums

Deterministic solution

- 25 times perform the following operation.
- Take $7 \cdot 10^6$ subsets of size 3.
- Sort them and take the two with the smallest difference (this difference is less than $\frac{3 \cdot 10^{12}}{7 \cdot 10^6} < 5 \cdot 10^5$).
- Remove these 6 values from the initial set.
- Now, we have 25 differences less than $5 \cdot 10^5$. Since, $25 \cdot 5 \cdot 10^5 < 2^{25}$, there exist subsets with equal sum.

D. Runs

Problem statement

- You are given the string.
- Run is a maximal sequence of contiguous equal characters.
- You have to calculate the number of strings with the same characters as given string and with the same number of runs.

D. Runs

Solution

- We will add symbols from 'a' to 'z' and calculate $dp[c][r]$ — the number of strings with the number of runs r after adding c first symbols.
- Suppose, we know the answer for first $c - 1$ symbols and we add symbol c.
- Let us insert into string with r runs. There are two different types of insert positions: $r + 1$ "in-between", giving 1 additional run, and $M - r$ "in-run", giving 2 additional runs.
- Fix a "in-between" and b "in-run". We add to $dp[c][r + a + 2 \cdot b]$ value $\binom{r+1}{a} \cdot \binom{M-r}{b} \cdot \binom{N_c-1}{a+b-1} \cdot dp[c-1][r]$.

E. Number Game

Problem statement

- There is a game. Each step: $(A, B) \to (A - k \cdot B, B)$ or $(A, B) \to (A, B - k \cdot A)$.
- The first one who gets zero or below loses.

E. Number Game

Solution

- Implement a naive solution, notice that all states $(A, B) A \geq B$ are winning if $A \geq B(A)$.
- $B(A)$ is also increasing.
- When $A \geq 2 \cdot B$ (A, B) is a winning position, because we could get to $(A \, mod \, B, B)$ or $(A \, mod \, B + B, B)$.
- At least one of those positions is losing.
- Otherwise, you can check if it's winning or losing recursively in $O(\log(n))$ steps.
- Use two pointers technique to determine $B(A)$.
- The answer could be calculated simply by iterating though all A in $[A_1, A_2]$.

F. Your Rank is Pure

Problem statement

- Number in S is considered **pure** if starting from it, you can continue taking its rank in S, and get a number in S, until in finite steps you hit 1.
- Find the number of ways to pick S with pure n from $\{1, \ldots, n\}$.

F. Your Rank is Pure

Solution

- Let us calculate $dp[n][k]$ — the number of subsets of size k where the biggest value is n and it is pure.
- $dp[n][k] = \sum_{i=1}^{k-1} \binom{n-k-1}{k-i-1} \cdot dp[k][i]$.
- Note, that you have to calculate dp one time for all the test cases.

G. Different Sum

Problem statement

- Cryptarithm equation is an addition equation in which all digits in one column are different.
- You need to calculate the number of cryptarithm equations with sum N in the given base B.

G. Different Sum

Solution

- First, calculate an auxiliary DP:
- $count[k][v][f]$ — the number of ways to represent v as a sum of k different non-negative numbers less than B. f is true, is there's a zero among those K numbers.
- Note that $v \leq B^2$.
- The main solution builds cryptarithm equations starting from the least significant digits.

G. Different Sum

Solution

- $dp[i][v][k][F]$ — the number of cryptarithm equations on the last i digits, such that its sum conicides with i last digits of N, carry is v, is a from filled columns, k is a number of non-finished rows and F is a boolean flag indicating is there was a zero in last column.
- The set of these parameters is enough to make a DP transition.
- For the next column we could choose the number of rows r that will be continued, how much we will add to get N_i, and if this column contains 0 or not.
- Multiply by some binomial coefficients and factorials.

H. Doubly-sorted grid

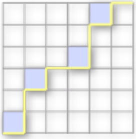
Problem statement

- Fill a rectangular grid with letters.
- Letters in each row and column should increase.

H. Doubly-sorted grid

Problem statement

- Consider the area containing all symbols $\leq c$
- It is a **Young diagram** for every c
- There are $\binom{2n}{n} \sim \frac{4^n}{\sqrt{n}}$ monotone paths from $(0, n)$ to $(n, 0)$



H. Doubly-sorted grid

Problem statement

- $dp[P][c][k]$ is the number of ways to fill a Young diagram P with symbols up to c and c only appears in columns $\leq k$.
- Add one more column and recalculate the path P.
- Generate all monotone paths before calculating DP and map them into integers.

I. The Reverse Problem About The Turtle

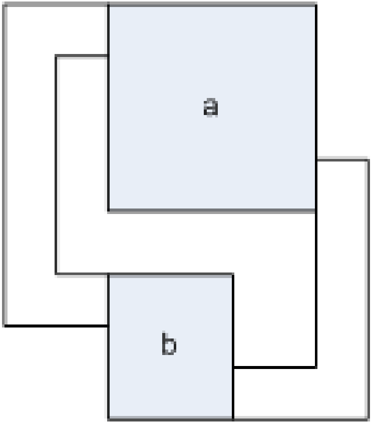
Problem statement

You have to build a grid such that the number of down-right paths from upper-left corner to lower-right corner is k.

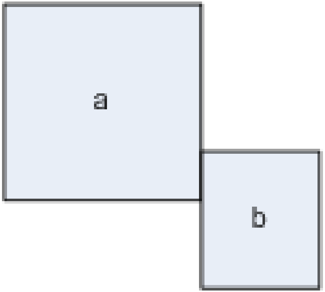
I. The Reverse Problem About The Turtle

Solution

The number of paths a and b to get $a + b$.



The number of paths a and b to get $a \cdot b$.



I. The Reverse Problem About The Turtle

Solution

- We get the answers for powers of 2 applying the first constuction.
- Decompose N on a powers of 2 and apply the second construction.
- Example for 19.