# Kingdom Partition

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Let us start by writing down a matrix of coefficients applied to the cost of an edge in the resulting functional depending on possible part belonging of its endpoints:

| | $A$ | $B$ | $C$ |
|---|---|---|---|
| $A$ | 2 | 0 | 1 |
| $B$ | 0 | 2 | 1 |
| $C$ | 1 | 1 | 0 |

Table 1. Desired coefficient matrix

The shape of the problem hints that it is somehow related to minimum cut, but the standard minimum cut-driven technique expresses the division of vertices into two parts, while we are asked about dividing vertices into three parts $A$, $B$ and $C$. The main trick is to perform a frequently appearing skew-symmetric transformation of a graph. Replace each vertex $v$ with two vertices $v_1$ and $v_2$ and each edge $uv$ of cost $l$ with two edges $u_1v_2$ and $u_2v_1$ of the same cost $l$.

Consider an arbitrary cut $(S, T)$ of a new graph into two disjoint vertex sets $S$ and $T$. Each vertex $v$ of the original graph may be seen as being in one of four states $SS$, $ST$, $TS$ and $TT$ depending on whether each of $v_1$ and $v_2$ belongs to $S$ or $T$. Write down a similar matrix of possible values of the coefficient applied to the cost of an edge (of the original graph) in the cut value $cut(S, T)$:

| | $ST$ | $TS$ | $SS$ | $TT$ |
|---|---|---|---|---|
| $ST$ | 2 | 0 | 1 | 1 |
| $TS$ | 0 | 2 | 1 | 1 |
| $SS$ | 1 | 1 | 0 | 2 |
| $TT$ | 1 | 1 | 2 | 0 |

Table 2. Coefficient matrix from cuts in a skew-symmetric graph

Note that the desired matrix is a submatrix of the matrix above. This "coincidence" hints that we must restrict vertex $a$ to be an $ST$-vertex and vertex $b$ to be a $TS$-vertex. Note that this can be done by connecting a source vertex $s$ with $a_1$ and $b_2$, and connecting $a_2$ and $b_1$ with a sink vertex $t$ using edges of infinite capacity and considering $s - t$ cuts in the resulting graph.

The last remaining issue is that we have distinct classes of $SS$ and $TT$ vertices. It turns out that the minimum cut may always be chosen such that there are no $TT$ vertices; indeed, make all $TT$ vertices be $SS$ vertices. As a result, no edge coefficient would increase; moreover, some $SS - TT$ edge coefficients would become $SS - SS$ edges, decreasing their coefficients from 2 to 0[1].

Combining everything together, we get a solution that constructs a skew-symmetric graph, finds a minimum cut in it using any appropriate maximum flow algorithm (e.g. Dinic algorithm), and then recovers the desired partition as $A = ST$, $B = TS$ and $C = SS \cup TT$.

[1]An educational remark. The last result is a special case of a generic *cut function submodularity* property: if $cut(X)$ is a value of the cut between $X$ and $V \setminus X$, then

$$cut(X \cap Y) + cut(X \cup Y) \le cut(X) + cut(Y).$$

Now apply this property to $X := S$ and $Y := T'$ where $T'$ is a set of vertices symmetric to the vertices in $T$ (i.e. $v_1' = v_2$ and $v_2' = v_1$), and obtain the previous result.