# Budget Distribution

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Let's first solve a single-topic problem. If there are no items that have too much assigned money (more they should have at the end, including unassigned money), the answer is 0. Otherwise, it's not important to which item money is assigned, if not giving too much money to any of the items. So, with a fixed set of items already having too much money, non-optimality function is linear-fractional in terms of extra money, i.e it has the form $\frac{ax+b}{cx+d}$. Also, it is easy to check that the derivative is increasing (it's negative and getting closer to zero), so overall the function is convex piecewise-linear-fractional function.

To solve the full problem, we need to find a way to combine solutions for different topics. Let $f_i$ be non-optimality function for $i$-th topic. In fact, we need to find the function $f(x) = \min_{\sum x_i = x, x_i \geq 0} f_i(x_i)$. As would be shown later, this function is convex piecewise-linear-fractional too, with a linear number of pieces.

To make a set of $x_i$ locally-optimal, it should be impossible to find a pair of indices $i$ and $j$, such that decreasing $x_i$ a bit, and increasing $x_j$ a bit would decrease the function. In fact, this means that all left derivatives should be less (greater by absolute value, as they are negative), than all right derivatives. Intuitively, if we think about the continuous process of distributing money, it should work like this: give next infinitely small amount of money to all topics with smallest derivatives (those, who are decreasing the fastest), in such proportion that their derivatives would be still the same. And the derivative of the resulting function in that point is equal to that minimum of the right derivatives.

So, let's construct a function by sweeping a line over its derivative. Each piece of the topics' function is working as "at this range of the derivatives, you also need to add this function". We know derivatives at each point, where switch from one piece to another happens, and the piece has a unique point with such derivative. So the only remaining part is to explicitly find the function for each part, when the set of topics we are giving money to is known.

For convenience, let $f_i(x) = \frac{a_i^2}{x-b_i} + c_i$. Any linear-fractional function with negative derivative can be written in such form. Intuitively, $b_i$ and $c_i$ are coordinates of the hyperbola center, and $a_i$ is just decreasing speed factor. This form is convenient, as $b_i$ and $c_i$ can be just thrown away, as $c_i$ only shifts resulting value by constant, and $b_i$ only shifts argument by constant. So we can find the result for functions of the form $\frac{a_i^2}{x}$, and then just shift it, to make value and derivative in a point where it's connected to the previous part correct.

For these kinds of functions, making derivatives same, means that $\frac{x_i}{x_j} = \frac{a_i}{a_j}$ for all pairs of $i$ and $j$. That means $x_i = \frac{a_i}{\sum a_i} x$. And the total function is $\frac{(\sum a_i)^2}{x}$.