Cactus without Bridges

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Let's define the following condition from the problem **"the length of each odd cycle is greater than or equal to the number of odd cycles"** as * condition.

Let's also define two types of cactuses:

- *Type 1* — The number of cycles of odd length is even;
- *Type 2* — The number of cycles of odd length is odd.

Let's prove that when the cactus is of *Type 2*, then it's impossible to give labeling. Let's suppose, to the contrary, we can give the labeling satisfying the conditions of the problem. Since the degree of each vertex of the graph is even and for each vertex, the labels should make an interval of integers, we can state that for each vertex v, the number of odd labels of the edges incident to the vertex v is equal to the number of even labels of the edges incident to the vertex v. Let's calculate the number of edges with odd labels by Handshaking lemma and we will get $\frac{E(G)}{2}$, which states that the number of edges in the graph should be even.

We will prove that for cactuses of *Type 1*, it's always possible to give labeling when * condition is available. Let's prove the following two facts:

- If the cactus is of *Type 1* and * condition is available, then we can label the edges of the graph satisfying the problem conditions;
- If the cactus is of *Type 2* and * condition is available, then we can label the edges of the graph satisfying problem conditions except for one arbitrary vertex u so that there will exist an integer x such that the labels of edges incident to the vertex u, x and $x + 2$ will form an interval of integers. More formally for the vertex u we have two missed integers for a full interval, and the difference between them is equal to 2.

First, let's note that * condition is satisfied for any cactus subgraph of the given cactus. We can prove one of these two facts, the other one will be proved in the same way. Let's prove the first one.

We will give proof using an induction method on the number of vertices of the graph. Let's take an arbitrary cycle from the given cactus, and let's consider that the cycle has the length k and the vertices of the cycle are u_1, u_2, $\ldots$, u_k. If we remove from the graph the edges of this cycle, we will have k components: G_1, G_2, $\ldots$, G_k, where G_i will contain the vertex u_i for each $1 \le i \le k$.

Now let's define the following **F** function for the vertices of the cycle in the following way:

- $|F(v_i)| = 1$, if G_i is of *Type 1* for $1 \le i \le k$;
- $|F(v_i)| = 2$, if G_i is of *Type 2* for $1 \le i \le k$;

where $|x|$ means an absolute value of x.

Let's define by c_1 — the number of i such that G_i is of *Type 1*, while by c_2 — the number of i such that G_i is of *Type 2* for $1 \le i \le k$.

If we label the edges of the cycle such that the difference between the labels of two consecutive edges will be equal to $F(w)$, where w is the common vertex of those two edges, then everything will be fine (we can shift the label of subgraphs in such a way that all conditions will be satisfied). Let's note that it's possible to label the edges of this cycle if the following condition $\sum_{i=1}^{k} F(v_i) = 0$ is true.

Now let's consider two cases:

Case 1: k is even. Here both c_1 and c_2 should be even, so we can take the values of F in the following way:

- The number of v_i such that $F(v_i) = 1$ is equal to $\frac{c_1}{2}$ for $1 \le i \le k$;
- The number of v_i such that $F(v_i) = -1$ is equal to $\frac{c_1}{2}$ for $1 \le i \le k$;

- The number of v_i such that $F(v_i) = 2$ is equal to $\frac{c_2}{2}$ for $1 \leq i \leq k$;
- The number of v_i such that $F(v_i) = -2$ is equal to $\frac{c_2}{2}$ for $1 \leq i \leq k$.

Case 2: k is odd. Here c_2 should be odd, while c_1 should be even. Since we have * condition, then you can note that $c_1 \geq 2$, so we can take the values of F in the following way:

- The number of v_i such that $F(v_i) = 1$ is equal to $\frac{c_1-2}{2}$ for $1 \leq i \leq k$;
- The number of v_i such that $F(v_i) = -1$ is equal to $\frac{c_1+2}{2}$ for $1 \leq i \leq k$;
- The number of v_i such that $F(v_i) = 2$ is equal to $\frac{c_2+1}{2}$ for $1 \leq i \leq k$;
- The number of v_i such that $F(v_i) = -2$ is equal to $\frac{c_2-1}{2}$ for $1 \leq i \leq k$.