Legacy Screensaver

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The rectangles overlap if and only if their individual projections on the x-axis and the y-axis both overlap. Thus, we can solve the problem separately for the axes, and then merge the results in some way.

Consider the x-axis, and imagine that instead of two rectangles, we just have two segments of lengths w_1 and w_2 moving inside the $[0; W]$ range. Each segment is moving at a speed of -1 or 1 and reflecting off the ends of the range.

It can be seen that the first segment returns to the same state (position and speed) after exactly $2(W - w_1)$ time units. Similarly, the period of movement of the second segment is $2(W - w_2)$.

If we look at the rectangles as a pair, the period of their common movement is $t_x = \operatorname{lcm}(2(W - w_1), 2(W - w_2))$, where $\operatorname{lcm}(p, q)$ stands for the least common multiple of p and q. That is, at any time τ, the segments overlap if and only if they also overlap at times $\tau + t_x, \tau + 2t_x, \ldots$.

We can write down a bit string a of size t_x, where a_τ is 1 if the segments overlap at time τ, and 0 otherwise. Similarly, we can write down a bit string b of size t_y, containing the same information for the projections of the rectangles on the y-axis.

Now, for any time τ, we can see that the rectangles overlap if and only if $a_{\tau \bmod t_x} = b_{\tau \bmod t_y} = 1$.

Consider two bits a_i and b_j $(0 \le i < t_x; 0 \le j < t_y)$. Does there exist time τ such that a_i will be matched against b_j — that is, such that $\tau \bmod t_x = i$ and $\tau \bmod t_y = j$? It is well-known that a solution τ to this system of modular equations exists if and only if $i \equiv j \pmod{g}$, where $g = \gcd(t_x, t_y)$ is the greatest common divisor of t_x and t_y.

Finally, it can be shown that all pairs (a_i, b_j) which satisfy $i \equiv j \pmod{g}$ will be matched against each other with equal frequencies. For each $r = 0, 1, \ldots, g - 1$, we can find how many indices i exist such that $a_i = 1$ and $i \bmod g = r$; let this number be c_r. Similarly, we can find how many indices j exist such that $b_j = 1$ and $j \bmod g = r$; let this number be d_r. Now, the numerator of the answer fraction is $\sum_{r=0}^{g-1} c_r \cdot d_r$, and the denominator is $\operatorname{lcm}(t_x, t_y)$. Don't forget to reduce the fraction.

Time complexity of this solution is $O((W + H)^2)$.