Incompetent Delivery Guy

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Let us call a vertex a *k-vertex* if an orc of incompetence k, starting from this vertex, will definitely reach Orthanc, but an orc of incompetence $k+1$ is not guaranteed to do that. (Or -1-vertex if an orc with any level of incompetence will not be able to reach Orthanc; or ∞-vertex if an orc with any finite level of incompetence will eventually reach Orthanc.) This unique number k can be called the *type* of this vertex, and the problem essentially consists of finding the type of the entrance vertex. We will present an algorithm of finding the types of all vertices.

It can be derived from the statement that -1-vertices are exactly the vertices from which there does not exist an oriented path to Orthanc. In particular, an orc succeeds with their order if and only if they never visit -1-vertices. Can we then characterize the 0-vertices? If a vertex (which is not a -1-vertex) has an arc from it to a -1-vertex, it is definitely a 0-vertex, but it is not a necessary condition. The necessary and sufficient condition is that on each shortest path from a vertex (which is not a -1-vertex) to Orthanc there is at least one vertex from which there is an arc to a -1-vertex.

Formally, let us define an *immediate k-vertex* as a k-vertex from which there is an arc to a $k-1$-vertex. Then, we can find all immediate k-vertices as the vertices from which there is an arc to at least one $k-1$-vertex, but which themselves are not ℓ-vertices for $\ell \leq k-1$. And k-vertices can be determined as the vertices from which any shortest path to Orthanc lies through at least one immediate k-vertex. Indeed, if a vertex is a k-vertex, then, no matter which shortest path Saruman chooses for an orc of incompetence $k+1$ in this vertex, at some moment they can hang around to a $k-1$ vertex, which means that they necessarily visit an immediate k-vertex. However, if an orc of incompetence at most k appears in this vertex, after the first hanging around they will appear in a vertex of type at least $k-1$. The converse (that if our method reported a vertex as a k-vertex then it is actually a k-vertex) can be proven similarly. Note that everything aforementioned does not apply to Orthanc, which is neither a k-vertex nor an immediate k-vertex for any finite k, it is always a ∞-vertex.

Hence, the solution is as follows. Firstly, we find $\overrightarrow{\mathrm{dist}}(u, O)$ for all vertices u using Dijkstra's algorithm ($\mathcal{O}(n + m \log n)$ is fast enough). In particular, we determine all -1-vertices. Then, for each $k = 0, 1, 2, \ldots$ we do the k^{th} stage of the algorithm. During this stage, we firstly find all immediate k-vertices simply by iterating over all ingoing edges in $k-1$-vertices. Then we can find all k-vertices using an algorithm similar to retroanalysis. This part works in $\mathcal{O}(m)$ time. The overall complexity is $\mathcal{O}(n + m \log n)$.